Characterization of the spectrum of irregular boundary value problem for the Sturm-Liouville operator

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Abstract. We consider the spectral problem generated by the Sturm-Liouville equation with an arbitrary complex-valued potential $q(x) \in L_2(0,\pi)$ and irregular boundary conditions. We establish necessary and sufficient conditions for a set of complex numbers to be the spectrum of such an operator.

In the present paper, we consider the eigenvalue problem for the Sturm-Liouvulle equation

$$u'' - q(x)u + \lambda u = 0 \tag{1}$$

on the interval $(0,\pi)$ with the boundary conditions

$$u'(0) + (-1)^{\theta} u'(\pi) + bu(\pi) = 0, \quad u(0) + (-1)^{\theta+1} u(\pi) = 0, \tag{2}$$

where b is a complex number, $\theta = 0, 1$, and the function $q(x)$ is an arbitrary complex-valued function of the class $L_2(0,\pi)$.

Denote by $c(x,\mu), s(x,\mu)$ $(\lambda = \mu^2)$ the fundamental system of solutions to (1) with the initial conditions $c(0,\mu) = s'(0,\mu) = 1$, $c'(0,\mu) = s(0,\mu) = 0$. The following identity is well known

$$c(x,\mu)s'(x,\mu) - c'(x,\mu)s(x,\mu) = 1. \tag{3}$$

Simple calculations show that the characteristic equation of (1), (2) can be reduced to the form $\Delta(\mu) = 0$, where

$$\Delta(\mu) = c(\pi,\mu) - s'(\pi,\mu) + (-1)^{\theta+1} b s(\pi,\mu). \tag{4}$$

The characteristic determinant $\Delta(\mu)$ of problem (1), (2), given by (4), is referred to as the characteristic determinant corresponding to the triple $(b, \theta, q(x))$. Throughout the following the symbol $||f||$ stands for

$||f||_{L_2(0,\pi)}$, $<q>=\frac{1}{\pi}\int_0^\pi q(x)dx$. By $\Gamma(z,r)$ we denote the disk of radius r centered at a point z. By PW_σ we denote the class of entire functions $f(z)$ of exponential type $\le\sigma$ such that $||f(z)||_{L_2(R)}<\infty$, and by PW_σ^- we denote the set of odd functions in PW_σ.

The following two assertions provide necessary and sufficient conditions to be satisfied by the characteristic determinant $\Delta(\mu)$.

Theorem 1. *If a function $\Delta(\mu)$ is the characteristic determinant corresponding to the triple $(b,\theta,q(x))$, then*

$$\Delta(\mu)=(-1)^{\theta+1}b\frac{\sin\pi\mu}{\mu}+\frac{f(\mu)}{\mu},$$

where $f(\mu)\in PW_\pi^-$.

Proof. Let $e(x,\mu)$ be a solution to (1) satisfying the initial conditions $e(0,\mu)=1$, $e'(0,\mu)=i\mu$, and let $K(x,t)$, $K^+(x,t)=K(x,t)+K(x,-t)$, and $K^-(x,t)=K(x,t)-K(x,-t)$ be the transformation kernels [1] that realize the representations

$$e(x,\mu)=e^{i\mu x}+\int_{-x}^{x}K(x,t)e^{i\mu t}dt,$$

$$c(x,\mu)=\cos\mu x+\int_0^x K^+(x,t)\cos\mu t dt,$$

$$s(x,\mu)=\frac{\sin\mu x}{\mu}+\int_0^x K^-(x,t)\frac{\sin\mu t}{\mu}dt. \tag{5}$$

It was shown in [2] that

$$c(\pi,\mu)=\cos\pi\mu+\frac{\pi}{2}<q>\frac{\sin\pi\mu}{\mu}-\int_0^\pi\frac{\partial K^+(\pi,t)}{\partial t}\frac{\sin\mu t}{\mu}dt, \tag{6}$$

$$s'(\pi,\mu)=\cos\pi\mu+\frac{\pi}{2}<q>\frac{\sin\pi\mu}{\mu}+\int_0^\pi\frac{\partial K^-(\pi,t)}{\partial x}\frac{\sin\mu t}{\mu}dt. \tag{7}$$

Substituting the right-hand sides of expressions (5), (6), (7) into (4), we obtain

$$\Delta(\mu) = (-1)^{\theta+1} b \tfrac{\sin \pi\mu}{\mu} +$$
$$+\tfrac{1}{\mu}\int_0^\pi [-\tfrac{\partial K^+(\pi,t)}{\partial t} - \tfrac{\partial K^-(\pi,t)}{\partial x} + (-1)^{\theta+1} b K^-(\pi,t)] \sin \mu t dt.$$

This relation, together with the Paley-Wiener theorem implies the assertion of Theorem 1.

Theorem 2. *Let a function $u(\mu)$ have the form*

$$u(\mu) = (-1)^{\theta+1} b \frac{\sin \pi\mu}{\mu} + \frac{f(\mu)}{\mu}, \tag{8}$$

where $f(\mu) \in PW_\pi^-$, b is a complex number. Then, there exists a function $q(x) \in L_2(0,\pi)$ such that the characteristic determinant correponding to the triple $(b, \theta, q(x))$ satisfies $\Delta(\mu) = u(\mu)$.

Proof. Since [3]

$$|f(\mu)| \le C_1 ||f(\mu)||_{L_2(R)} e^{\pi|Im\mu|}, \tag{9}$$

it follows that there exists an arbitrary large positive integer N such that

$$|u(\mu)| < 1/10 \tag{10}$$

on the set $|Im\mu| \le 1$, $Re\mu \ge N$. Let μ_n $(n = 1, 2, \ldots)$ be a strictly monotone increasing sequence of positive numbers such that $|\mu_n - (N + 1/2)| < 1/10$ if $1 \le n \le N$ and $\mu_n = n$ if $n \ge N+1$. Consider the function

$$s(\mu) = \pi \prod_{n=1}^{\infty} \frac{\mu_n^2 - \mu^2}{n^2} = \frac{\sin \pi\mu}{\mu} \prod_{n=1}^{N} \frac{\mu_n^2 - \mu^2}{n^2 - \mu^2}. \tag{11}$$

Obviously, all zeros of the function $s(\mu)$ are simple, and, in addition, the inequality

$$(-1)^n \dot{s}(\mu_n) > 0. \tag{12}$$

holds for any n. It was shown in [4] that

$$\dot{s}(n) = \frac{\pi(-1)^n}{n}(1 + C_0 n^{-2} + O(n^{-4})), \tag{13}$$

where C_0 is some constant, and the asymptotic formula

$$s(\mu) = \frac{\sin \pi\mu}{\mu} + O(\mu^{-3}) \tag{14}$$

holds in the strip $|Im\mu| \leq 1$.

Consider the equation

$$z^2 - u(\mu_n)z - 1 = 0. \tag{15}$$

It has the roots

$$c_n^{\pm} = \frac{u(\mu_n) \pm \sqrt{u^2(\mu_n) + 4}}{2}. \tag{16}$$

It follows from (10) that for any n all numbers c_n^+ lie in the disk $\Gamma(1, 1/2)$ and all numbers c_n^- lie in the disk $\Gamma(-1, 1/2)$. Let for even n $c_n = c_n^+$, and for odd n $c_n = c_n^-$. Then $(-1)^n Rec_n > 0$ for any $n = 1, 2, \ldots$. This, together with (12) implies that $Rew_n > 0$ for any n, where

$$w_n = \frac{c_n}{\mu_n \dot{s}(\mu_n)}. \tag{17}$$

We set $F(x,t) = F_0(x,t) + \hat{F}(x,t)$, where

$$F_0(x,t) = \sum_{n=1}^{N} \left(\frac{2c_n}{\mu_n \dot{s}(\mu_n)} \sin \mu_n x \sin \mu_n t - \frac{2}{\pi} \sin nx \sin nt \right),$$

$$\hat{F}(x,t) = \sum_{n=N+1}^{\infty} \left(\frac{2c_n}{\mu_n \dot{s}(\mu_n)} \sin \mu_n x \sin \mu_n t - \frac{2}{\pi} \sin nx \sin nt \right). \tag{18}$$

One can readily see that $F_0(x,t) \in C^\infty(R^2)$. Consider the function $\hat{F}(x,t)$. If $n \geq N+1$, then, by taking into account (9), (16) and the rule for choosing the roots of equation (15), we obtain

$$c_n = (-1)^n + \frac{f(n)}{2n} + O(1/n^2). \tag{19}$$

It follows from (9), (13), (18), and (19) that

$$\hat{F}(x,t) = \textstyle\sum_{n=N+1}^{\infty} \frac{2}{\pi} \left(\frac{1+(-1)^n \frac{f(n)}{2n} + O(1/n^2)}{1+c_0/n^2+O(1/n^4)} - 1 \right) \sin nx \sin nt =$$
$$= \textstyle\sum_{n=N+1}^{\infty} \frac{2}{\pi} [(1+(-1)^n \frac{f(n)}{2n} + O(1/n^2)) \times$$
$$(1 - c_0/n^2 + O(1/n^4)) - 1] \sin nx \sin nt =$$
$$= \tfrac{2}{\pi} \textstyle\sum_{n=N+1}^{\infty} ((-1)^n \frac{f(n)}{2n} + O(1/n^2)) \sin nx \sin nt =$$
$$= (\hat{G}(x-t) - \hat{G}(x+t))/2,$$

where

$$\hat{G}(y) = \frac{2}{\pi} \sum_{n=N+1}^{\infty} ((-1)^n \frac{f(n)}{2n} + O(1/n^2)) \cos ny.$$

The relation

$$\sum_{n=1}^{\infty} |f(n)|^2 = \frac{1}{2} ||f(\mu)||_{L_2(R)},$$

which follows from the Paley-Wiener theorem, together with the Parseval equality, implies that $\hat{G}(y) \in W_2^1[0, 2\pi]$. Therefore, we obtain the representation

$$F(x,t) = F_0(x,t) + (\hat{G}(x-t) - \hat{G}(x+t))/2, \tag{20}$$

where the functions $F_0(x,t)$ and $\hat{G}(y)$ belong to the above-mentioned classes.

Now let us consider the Gelfand-Levitan equation

$$K(x,t) + F(x,t) + \int_0^x K(x,s) F(s,t) ds = 0 \tag{21}$$

and prove that it has a unique solution in the space $L_2(0,x)$ for each $x \in [0,\pi]$. To this end, it suffices to show that the corresponding homogeneous equation has only the trivial solution.

Let $f(t) \in L_2(0,x)$. Consider the equation

$$f(t) + \int_0^x F(s,t) f(s) ds = 0.$$

Following [4], by multiplying the last equation by $\bar{f}(t)$ and by integrating the resulting relation over the interval $[0,x]$, we obtain

$$\int_0^x |f(t)|^2dt + \sum_{n=1}^{\infty} \frac{2c_n}{\mu_n \dot{s}(\mu_n)} \int_0^x \bar{f}(t)\sin\mu_n t dt \int_0^x f(s)\sin\mu_n s ds - \\ - \sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^x \bar{f}(t)\sin nt dt \int_0^x f(s)\sin ns ds = 0.$$

This, together with the Parseval equality for the function system $\{\sin nt\}_1^{\infty}$ on the interval $[0,\pi]$ implies that

$$\sum_{n=1}^{\infty} w_n \left| \int_0^x f(t)\sin\mu_n t dt\right|^2 = 0,$$

where the w_n are the numbers given by (17). Since $Re w_n > 0$, we see that $\int_0^x f(t)\sin\mu_n t dt = 0$ for any $n = 1, 2, \ldots$. Since [5, 6] the system $\{\sin\mu_n t\}_1^{\infty}$ is complete on the interval $[0,\pi]$, we have $f(t) \equiv 0$ on $[0,x]$.

Let $\hat{K}(x,t)$ be a solution of equation (21), and let $\hat{q}(x) = 2\frac{d}{dx}\hat{K}(x,x)$; then it follows [4] from (20) that $\hat{q}(x) \in L_2(0,\pi)$. By $\hat{s}(x,\mu)$, $\hat{c}(x,\mu)$ we denote the fundamental solution system of equation (1) with potential $\hat{q}(x)$ and the initial conditions $\hat{s}(0,\mu) = \hat{c}'(0,\mu) = 0$, $\hat{c}(0,\mu) = \hat{s}'(0,\mu) = 1$. By reproducing the corresponding considerations in [4], we obtain $\hat{s}(\pi,\mu) \equiv s(\mu)$, whence it follows that the numbers μ_n^2 form the spectrum of the Dirichlet problem for equation (1) with potential $\hat{q}(x)$, and $\hat{c}(\pi,\mu_n) = c_n$, which, together with identity (3), implies that $\hat{s}'(\pi,\mu_n) = 1/c_n$.

Let $\hat{\Delta}(\mu)$ be the characteristic determinant corresponding to the triple $(b,\theta,\hat{q}(x))$. Let us prove that $\hat{\Delta}(\mu) \equiv u(\mu)$. By Theorem 1, the function $\hat{\Delta}(\mu)$ admits the representation

$$\hat{\Delta}(\mu) = (-1)^{\theta+1} b \frac{\sin\pi\mu}{\mu} + \frac{\hat{f}(\mu)}{\mu},$$

where $\hat{f}(\mu) \in PW_{\pi}^{-}$. By taking into account (4) and the fact that the numbers c_n are roots of equation (15), we have

$$\hat{\Delta}(\mu_n) = \hat{c}(\pi,\mu_n) - \hat{s}'(\pi,\mu_n) + (-1)^{\theta+1} b\hat{s}(\pi,\mu_n) = c_n - c_n^{-1} = u(\mu_n).$$

Hence it follows that the function

$$\Phi(\mu) = \frac{u(\mu) - \hat{\Delta}(\mu)}{s(\mu)} = \frac{f(\mu) - \hat{f}(\mu)}{\mu s(\mu)}$$

is an entire function on the complex plane. Since the function $g(\mu) = f(\mu) - \hat{f}(\mu)$ belongs to PW_π^-, it follows from (9) that

$$|g(\mu)| \le C_2 e^{\pi |Im\mu|}. \tag{22}$$

Relation (11) implies that if $|Im\mu| \ge 1$, then

$$|\mu s(\mu)| \ge C_3 e^{\pi |Im\mu|} \tag{23}$$

$(C_3 > 0)$. hence we obtain $|Im\mu| \ge 1$ if $|\Phi(\mu)| \le C_2/C_3$.

By H we denote the union of the vertical segments $\{z : |Rez| = n + 1/2, |Imz| \le 1\}$, where $n = N+1, N+2, \ldots$. It follows from (11) that if $\mu \in H$, then $|\mu s(\mu)| \ge C_4 > 0$. The last inequality, together with (22), (23), and the maximum principle for the absolute value of an analytic function, implies that $|\Phi(\mu)| \le C_5$ in the strip $|Im\mu| \le 1$. Consequently, the function $\Phi(\mu)$ is bounded on the entire complex plane and hence identically constant by the Liouville theorem. It follows from the Paley-Wiener theorem and the Riemann lemma [1] that if $|Im\mu| = 1$, then $\lim_{|\mu|\to\infty} g(\mu) = 0$, whence we obtain $\Phi(\mu) \equiv 0$.

The proof of Theorem 2 is complete.

Further we consider problem (1), (2) under the supplementary condition $b \ne 0$.

Theorem 3. *For a set Λ of complex numbers to be the spectrum of problem (1), (2) it is necessary and sufficient that it has the form $\Lambda = \{\lambda_n\}$, where $\lambda_n = \mu_n^2$,*

$$\mu_n = n + r_n,$$

where $\{r_n\} \in l_2$, $n = 1, 2, \ldots$.

Necessity. It follows from Theorem 1 that the characteristic equation of problem (1), (2) can be reduced to the form

$$(-1)^{\theta} b \frac{\sin \pi\mu}{\mu} = \frac{f(\mu)}{\mu}, \tag{24}$$

where $f(\mu) \in PW_{\pi}^{-}$. It was shown in [1] that equation (24) has the roots $\mu_n = n + r_n$, where $r_n = o(1)$, $n = 1, 2, \ldots$. Hence it follows that

$$\sin \pi r_n = (-1)^{\theta+n} f(n + r_n)/b.$$

Since $\{f(n + r_n)\} \in l_2$, by [4], it follows that $\{r_n\} \in l_2$.

Sufficiency. Let the set Λ admit the representation of the above-mentioned form. We denote

$$u(\mu) = (-1)^{\theta+1} b\pi\mu \prod_{n=1}^{\infty} \left(\frac{\lambda_n - \mu^2}{n^2} \right).$$

It follows from [10] and the conditions of the theorem that the infinite product in the right-hand side of the last equality converges uniformly in any bounded domain. We denote $\phi(\mu) = (-1)^{\theta+1} b \sin \pi\mu - u(\mu)$. Let us prove that $\phi(\mu) \in PW_{\pi}^{-}$. Evidently, $\phi(\mu)$ is an odd entire function. Obviously,

$$|Im\mu_n| \leq M,$$

where M is some constant. By Γ we denote the union of disks of radius 1/2 centered at the points $n = 1, 2, \ldots$. Let $\mu \notin \Gamma$; then it follows from well known identity

$$\sin \pi\mu = \pi\mu \prod_{n=1}^{\infty} \frac{n^2 - \mu^2}{n^2} \tag{25}$$

that

$$\phi(\mu) = \sin \pi\mu (1 - \phi_0(\mu)), \tag{26}$$

where

$$\phi_0(\mu) = \prod_{n=1}^{\infty} (1 + \alpha_n(\mu)), \tag{27}$$

where

$$\alpha_n(\mu) = \frac{\mu_n^2 - n^2}{n^2 - \mu^2}. \tag{28}$$

Let us study the function $\phi_0(\mu)$ for $Re\mu \geq 0$, $\mu \notin \Gamma$. One can readily see that

$$\begin{gathered} |\mu_n + n| < cn, \quad |n + \mu| > |\mu|, \\ |n - \mu| > 1/2, \quad |n + \mu| > n. \end{gathered} \tag{29}$$

It follows from (28) and (29) that

$$\begin{gathered} \sum_{n=1}^{\infty} |\alpha_n(\mu)| \leq c_1 \sum_{n=1}^{\infty} \frac{|r_n| n}{|n+\mu||n-\mu|} \leq \\ \leq c_1 \sum_{n=1}^{\infty} \frac{|r_n|}{|n-\mu|} \leq c_1 \left(\sum_{n=1}^{\infty} \frac{|r_n|^2}{|n-\mu|^{1/2}} + \sum_{n=1}^{\infty} \frac{1}{|n-\mu|^{3/2}}\right) \leq c_2. \end{gathered} \tag{30}$$

Obviously, the inequality $|r_n| < 1/(8c_1)$ holds for all $n > N$, where N is a sufficiently large number. It follows from (28) and (29) that for all $n > N$

$$|\alpha_n(\mu)| \leq 1/4. \tag{31}$$

Relations (30), (31) and the obvious inequality

$$|\ln(1 + z)| \leq 2|z|, \tag{32}$$

valid for $|z| \leq 1/4$ imply that

$$\sum_{n=N+1}^{\infty} |\ln(1 + \alpha_n(\mu))| \leq c_3,$$

moreover, here and throughout the following, we choose the branch of $\ln(1 + z)$ that is zero for $z = 0$. Now, by [7],

$$\prod_{n=N+1}^{\infty} |1 + \alpha_n(\mu)| \leq e^{c_3},$$

consequently,

$$\prod_{n=1}^{\infty} |1 + \alpha_n(\mu)| \leq c_4 e^{c_3}. \tag{33}$$

Since $\phi_(\mu)$ is an even function, it follows from (26), (27), and (33) that

$$|\phi(\mu)| \le c_5 e^{\pi|Im\mu|}. \tag{34}$$

The maximum principle implies that inequality (34) is valid in the entire complex plane, therefore, the function $\phi(\mu)$ is an entire function function of exponential type $\le \pi$. Evidently, $|r_n| < c_6$. Notice, that if $|Im\mu| \ge \tilde{C}$, where $\tilde{C} = 4c_1c_6$, then estimate (31) holds for any $n = 1, 2, \ldots$. In the domain $|Im\mu| \ge \tilde{C}$ we define a function

$$W(\mu) = \ln\phi_0(\mu) = \sum_{n=1}^{\infty} \ln(1 + \alpha_n(\mu)),$$

then we have

$$\phi(\mu) = \sin\pi\mu(1 - e^{W(\mu)}). \tag{35}$$

Let us estimate the function $W(\mu)$. One can readily see that

$$\lim_{|Im\mu|\to\infty} \left(\sum_{n=1}^{\infty} \frac{|r_n|^2}{|n-\mu|^{1/2}} + \sum_{n=1}^{\infty} \frac{1}{|n-\mu|^{3/2}} \right) = 0.$$

This, together with (30) and (32) implies that the inequality

$$|W(\mu)| \le \textstyle\sum_{n=1}^{\infty} |\ln(1 + \alpha_n(\mu))| \le 2\sum_{n=1}^{\infty} |\alpha_n(\mu)|) \le 1/4$$

holds in the domain $|Im\mu| \ge \hat{C}$, where $\hat{C}$ is a sufficiently large number. Therefore, it follows from the elementary inequality $|1 - e^z| \le 2|z|$, valid for $|z| \le 1/4$, that $|1 - e^{W(\mu)}| \le 2|W(\mu)|$, which, together with (35) implies that

$$|\phi(\mu)| \le c_7|W(\mu)|. \tag{36}$$

for $\mu \in l$, where l is the line $Im\mu = \hat{C}$. Let us prove the inequality

$$\int_l |W(\mu)|^2 d\mu < \infty. \tag{37}$$

Let $\mu \in l^+$, where l^+ is the ray $Im\mu = \hat{C}, Re\mu \ge 0$. The elementary inequality

$$|\ln(1+z) - z| \le |z|^2,$$

valid for $|z| \leq 1/2$, implies that

$$|W(\mu)| \leq |S_1(\mu)| + S_2(\mu),$$

where

$$S_1(\mu) = \sum_{n=1}^{\infty} \alpha_n(\mu), \quad S_2(\mu) = \sum_{n=1}^{\infty} |\alpha_n(\mu)|^2.$$

Set

$$I_m = \int_{l^+} |S_m(\mu)|^2 d\mu$$

$(m = 1, 2)$. First, consider the integral I_1. One can readily see that

$$I_1 = \int_{l^+} \left|\sum_{n=1}^{\infty} \frac{(\mu_n+n)r_n}{(n+\mu)(n-\mu)}\right|^2 d\mu \leq c_7(\int_{l^+} \left|\mu \sum_{n=1}^{\infty} \frac{r_n}{(n+\mu)(n-\mu)}\right|^2 d\mu + \\ + \int_{l^+} \left|\sum_{n=1}^{\infty} \frac{r_n}{(n+\mu)}\right|^2 d\mu + \int_{l^+} \left|\sum_{n=1}^{\infty} \frac{r_n^2}{(n+\mu)(n-\mu)}\right|^2 d\mu). \tag{38}$$

The convergence of the first and the third integrals in the right-hand side of (38) was establised in [8], By [9] so is the second integral. It is readily seen that

$$I_2 \leq c_8 \int_{l^+} \left|\sum_{n=1}^{\infty} \frac{|r_n|^2}{|n-\mu|^2}\right|^2 d\mu. \tag{39}$$

By [8] the integral in right-hand side of (39) is convergent.

From the last inequality, the convergence of the integral I_1 and the evenness of the function $W(\mu)$, we find that inequality (37) is valid. It follows from (36), (37) and [3] that

$$\int_R |\phi(\mu)|^2 d\mu < \infty,$$

consequently, $\phi(\mu) \in PW_\pi^-$, which, together with Theorem 2, proves theorem 3.

Consider problem (1), (2) if $b = 0$. Substituting the functions $c(x, \mu)$, $s(x, \mu)$ into boundary conditions and taking into account (3), we find that

each root subspace contains one eigenfunction and, possibly, associated functions. The characteristic equation has the form

$$\frac{f(\mu)}{\mu} = 0,$$

where $f(\mu) \in PW_\pi^-$. Let us consider two examples.

1). Set

$$f_1(\mu) = \frac{\sin^k(\alpha\pi\mu/k)\sin^k((1-\alpha)\pi\mu/k)}{\mu^{2k-1}},$$

where k is an arbitrary natural number, and α is an irrational number, $0 < \alpha < 1$. Obviously, $f_1(\mu) \in PW_\pi^-$. Then, by Theorem 2, there exists a potential $q_1(x) \in L_2(0,\pi)$, such that the corresponding characteristic determinant $\Delta_1(\mu) = f_1(\mu)/\mu$. Since the equations $\sin(\alpha\pi\mu/k) = 0$ and $\sin((1-\alpha)\pi\mu/k) = 0$ have no common roots, except zero, we see that each root subspace of problem (1), (2) with potential $q_1(x)$ contains one eigenfunction and associated functions up to order $k-1$. One can readily see that $|\Delta_1(\mu)| \geq ce^{|Im\mu|\pi}|\mu|^{1-2k}$ $(c > 0)$, if μ belongs to a sequence of infinitely expanding contours. Then, by [11], the system of eigen-and associated functions of problem (1), (2) is complete in $L_2(0,\pi)$.

2). Set $f(\mu) = \sin^2(\pi\mu/2)/\mu$. It follows from (25) that

$$f(\mu) = \frac{\pi^2}{4}\mu\prod_{n=1}^{\infty}\left(\frac{(2n)^2-\mu^2}{(2n)^2}\right)^2.$$

We denote

$$u(\mu) = \frac{\pi^2}{4}\mu\prod_{n=1}^{\infty}\left(\frac{\mu_n^2-\mu^2}{(2n)^2}\right)^2, \tag{40}$$

where $\mu_n = 2n$, if $n \neq 2^p + k$, $k = 1,\ldots,[\ln p]$, $p = p_0, p_0+1,\ldots$. $\mu_n = 2^{p+1}$, if $n = 2^p + k$, $k = 1,\ldots,[\ln p]$, $p = p_0, p_0+1,\ldots$ $(p_0 \geq 10)$. It can easily be checked that

$$\lim_{n\to\infty}\frac{\mu_n}{n} = 2, \quad 0 < c_1 < \prod_{n=1}^{\infty}\frac{\mu_n}{n} < \infty.$$

This, together with [7] implies that the infinite product in right-hand side of (40) uniformly convergents in any bounded domain of the complex plane, therefore, $u(\mu)$ is an entire analytical function.

Let us prove that $u(\mu) \in PW_{\pi}^{-}$. We set $\psi(\mu) = f(\mu) - u(\mu)$. By Γ we denote the union of disks of radius 1 centered at the points $2n$, $n = 1, 2, \ldots$. Let $\mu \notin \Gamma$, $Re\mu \geq 0$; then

$$\psi(\mu) = f(\mu)(1 - \phi(\mu)),$$

where

$$\phi(\mu) = (\textstyle\prod_{p=p_0}^{\infty} A_p(\mu))^2, \quad A_p(\mu) = \prod_{k=1}^{[\ln p]}(1 + \alpha_{p,k}(\mu)),$$
$$\alpha_{p,k}(\mu) = \tfrac{-(2^{p+1}+k)k}{(2^p+k-\mu/2)(2^p+k+\mu/2)}.$$

Trivially,

$$|\alpha_{p,k}(\mu)| \leq 2 \ln p. \tag{41}$$

Consider two cases. Let $|\mu/2 - 2^p| \geq 2^p/10$ for all $p = p_0, p_0 + 1, \ldots$; then

$$|\alpha_{p,k}(\mu)| \leq \frac{2 \ln p}{|2^p + k - \mu/2|} \leq 1/4. \tag{42}$$

Consider the function

$$F(\mu) = \sum_{p=p_0}^{\infty} \sum_{k=1}^{[\ln p]} \ln(1 + \alpha_{p,k}(\mu)).$$

It follows from (31), (41), and (42) that

$$|F(\mu)| = \textstyle\sum_{p=p_0}^{\infty} \sum_{k=1}^{[\ln p]} |\ln(1 + \alpha_{p,k}(\mu))| \leq \sum_{p=p_0}^{\infty} \sum_{k=1}^{[\ln p]} |\alpha_{p,k}(\mu))| \leq$$
$$\leq c_1 \textstyle\sum_{p=p_0}^{\infty} \frac{\ln^2 p}{|2^p - \mu/2|} \leq c_2 \sum_{p=p_0}^{\infty} \frac{\ln^2 p}{2^p} \leq c_3.$$

Now, by [7],

$$|\phi(\mu)| \leq e^{2c_3}. \tag{43}$$

Suppose for some $\tilde{p} \geq p_0$

$$|\mu/2 - 2^{\tilde{p}}| < 2^{\tilde{p}}/10. \tag{44}$$

Evidently, $\phi(\mu)=\gamma^2_{\tilde p}(\mu)\beta^2_{\tilde p}(\mu)$, where

$$\gamma_{\tilde p}(\mu)=\prod_{k=1}^{[\ln p]}(1+\alpha_{\tilde p,k}(\mu)),\quad \beta_{\tilde p}(\mu)=\prod_{p=p_0,p\neq\tilde p}^{\infty}\prod_{k=1}^{p}(1+\alpha_{p,k}(\mu))$$

Arguing as above, we see that

$$|\beta_{\tilde p}(\mu)|\leq c_4,$$

where c_4 does not depend of $\tilde p$. It follows from (41) and (44) that

$$|\gamma_{\tilde p}|^2\leq(1+2\ln\tilde p)^{2\ln\tilde p}\leq c_5 2^{\tilde p/3}\leq c_6|\mu|^{1/3}.$$

From this inequality and the evenness of the function $\phi(\mu)$, we find that the inequality

$$|\psi(\mu)|\leq|f(\mu)|(|\mu|+3)^{1/3}$$

is valid outside Γ. Since $|f(\mu)|\leq c_7/(|\mu|+3)$ in the strip $\Pi:|Im\mu|\leq 2$, we see that the relation

$$|\psi(\mu)|\leq c_8(|\mu|+3)^{-2/3}$$

holds on the set $\Pi\setminus\Gamma$. The last inequality and the maximum principle imply that

$$|\psi(\mu)|\leq c_8(|\mu|+1)^{-2/3}$$

in the strip Π, consequently, $\psi(\mu)\in PW^-_\pi$, hence, $u(\mu)\in PW^-_\pi$. Then by theorem 2 there exists a potential $q(x)\in L_2(0,\pi)$, such that the correponding characteristic determinant $\Delta(\mu)=u(\mu)/\mu$. This yields that the dimensions of root subspaces of problem (1), (2) with potential $q(x)$ increase infinitely, and the system of root functions contains associated functions of arbitrarily high order.

By Q_p we denote the disk $|\mu/2-2^p|<2^p/10$, $Q=\bigcup_{p=p_0}^{\infty}$, $D=\mathbb{C}\setminus(\Gamma\bigcup Q)$. In the domain D we consider the function $\tilde\phi(\mu)=f(\mu)/u(\mu)$. It is readily seen that

$$\tilde\phi(\mu)=(\textstyle\prod_{p=p_0}^{\infty}\tilde A_p(\mu))^2,\quad \tilde A_p(\mu)=\prod_{k=1}^{[\ln p]}(1+\tilde\alpha_{p,k}(\mu)),$$
$$\tilde\alpha_{p,k}(\mu)=\tfrac{(2^{p+1}+k)k}{(2^p-\mu/2)(2^p+\mu/2)}.$$

Arguing as above, we see that

$$|\tilde{\phi}(\mu)| \leq c_9.$$

This implies that $\Delta(\mu) \geq c_{10}e^{|Im\mu|\pi}/|\mu|^2$ ($c_{10} > 0$) if μ belongs to a sequence of infinitely expanding contours. Then, by [11], the system of eigen-and associated functions of problem (1), (2) is complete in $L_2(0,\pi)$.

Characterization of the spectrum of nonselfadjoint problem (1), (2) in the case of separated boundary conditions was given in [12], and for periodic and antiperiodic boundary conditions an analogous question was solved in [9].

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